

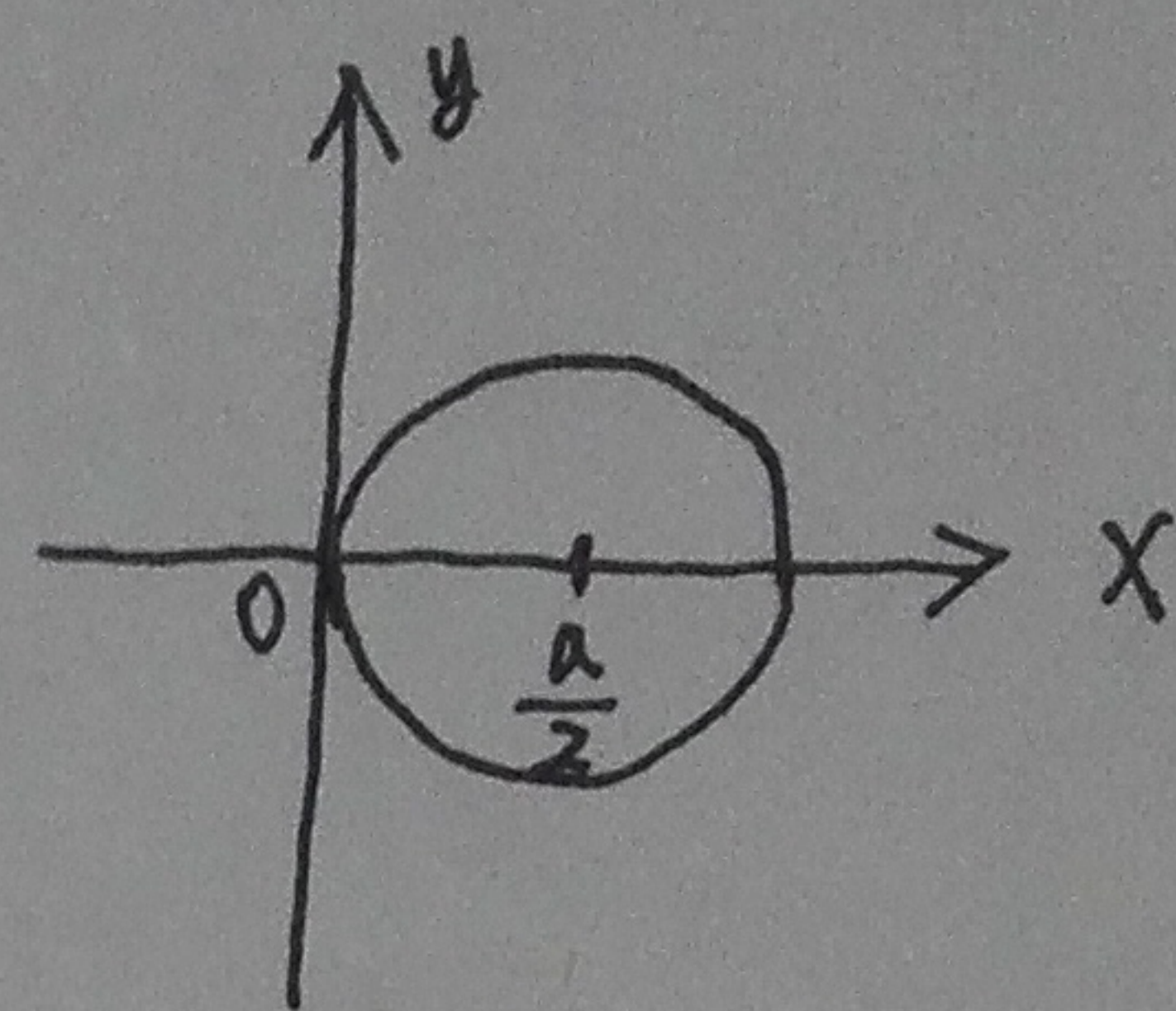
同步练习 P213 3.

廖国勇 2016. 4. 15

3. 求 $\oint_C \sqrt{x^2+y^2} ds$, 其中 C 是圆周 $x^2+y^2=ax$ ($a>0$)

解: (方法一), 设圆周上点的参数表示为

$$\begin{cases} x = \frac{a}{2} + \frac{a}{2} \cos \theta \\ y = \frac{a}{2} \sin \theta \end{cases} \quad 0 \leq \theta \leq 2\pi$$



$$\begin{aligned} \sqrt{x^2+y^2} &= \sqrt{\left(\frac{a}{2}\right)^2 [(1+\cos\theta)^2 + \sin^2\theta]} = \frac{a}{2} \sqrt{2+2\cos\theta} = \frac{a}{2} \sqrt{4\cos^2\frac{\theta}{2}} \\ &= a |\cos\frac{\theta}{2}| \end{aligned}$$

$$ds = \sqrt{(dx)^2 + (dy)^2} = \frac{a}{2} d\theta$$

$$\begin{aligned} \text{于是, } \oint_C \sqrt{x^2+y^2} ds &= \int_0^{2\pi} a |\cos\frac{\theta}{2}| \cdot \frac{a}{2} d\theta = a^2 \int_0^{\pi} \cos\frac{\theta}{2} d\theta \\ &= a^2 \cdot \frac{4}{2} \sin\frac{\theta}{2} \Big|_0^{\pi} = \frac{a^2}{2} \cdot 4 = 2a^2 \end{aligned}$$

(方法二). 设圆周上点的参数表示为

$$\begin{cases} x = \rho \cos \theta \\ y = \rho \sin \theta \end{cases}$$

圆周 $x^2+y^2=ax$ 化为 $\rho^2 = a\rho \cos \theta$, 即 $\rho = a \cos \theta$

这里 $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$. 显然 $\sqrt{x^2+y^2} = \sqrt{\rho^2} = \sqrt{a^2 \cos^2 \theta} = a |\cos \theta| = a \cos \theta$

$$ds = \sqrt{(\rho')^2 + \rho^2} d\theta = \sqrt{(a \sin \theta)^2 + (a \cos \theta)^2} d\theta = a d\theta$$

$$\begin{aligned} \text{于是, } \oint_C \sqrt{x^2+y^2} ds &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} a \cos \theta \cdot a d\theta = a^2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos \theta d\theta \\ &= 2a^2 \end{aligned}$$



同步练习 P213 4. 廖国勇 2016.4.15

4. 求 $\oint_L \sqrt{2y^2+z^2} ds$, 其中 L 是曲面 $x^2+y^2+z^2=a^2$ 与 $y=x$ 的交线

解: 利用球面坐标
$$\begin{cases} x = r \sin \varphi \cos \theta \\ y = r \sin \varphi \sin \theta \\ z = r \cos \varphi \end{cases} \quad \text{这里 } r=a (>0), \quad 0 \leq \theta \leq 2\pi \\ 0 \leq \varphi \leq \pi$$

$$\text{由于 } y=x \Rightarrow r \sin \varphi \cos \theta = r \sin \varphi \sin \theta \Rightarrow \theta = \frac{\pi}{4} \text{ 和 } \theta = \frac{5\pi}{4}$$

$$\text{在 } \theta = \frac{\pi}{4} \text{ 时, } \begin{cases} x = \frac{\sqrt{2}}{2} a \sin \varphi \\ y = \frac{\sqrt{2}}{2} a \sin \varphi \\ z = a \cos \varphi \end{cases}, \quad \sqrt{2y^2+z^2} = a$$

$$ds = \sqrt{(dx)^2 + (dy)^2 + (dz)^2} = a d\varphi$$

由对称性, 只要计算 $\theta = \frac{\pi}{4}$ 时的积分, 然后乘以 2 即可

$$\text{于是 } \oint_L \sqrt{2y^2+z^2} ds = 2 \int_0^\pi a \cdot a d\varphi = 2\pi a^2$$

注: 利用弧长积分的物理意义: L 构件的质量.

L 就是 ^{圆心为} 过原点, 半径为 a 的圆周, 其周长为 $2\pi a$

线密度函数 $f(x,y,z) = \sqrt{2y^2+z^2} = \sqrt{x^2+y^2+z^2} = a$. ($y=x$ 要注意)

即线密度为常数 a , 于是, $\oint_L \sqrt{2y^2+z^2} ds = f(x,y,z) \cdot \text{周长}$

$$= a \cdot 2\pi a = 2\pi a^2$$

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7. 求 $\int_C x \sin(xy) dx - y \cos(xy) dy$, 其中 C 是从点 $O(0,0)$ 到 $A(1,\pi)$ 的线段

解: 直线 OA 方程: $y = \pi x$, 于是

$$\begin{aligned} & \int_C x \sin(xy) dx - y \cos(xy) dy \\ &= \int_0^1 x \sin(x \cdot \pi x) dx - \pi x \cos(x \cdot \pi x) \cdot d(\pi x) \\ &= \int_0^1 [x \sin(\pi x^2) - \pi^2 x \cos(\pi x^2)] dx \\ &= \left(-\frac{1}{2\pi} \cos(\pi x^2) - \frac{\pi}{2} \sin(\pi x^2) \right) \Big|_0^1 \\ &= \left(\frac{1}{2\pi} - 0 \right) - \left(-\frac{1}{2\pi} - 0 \right) \\ &= \frac{1}{\pi} \end{aligned}$$

注: $\int x \sin(\pi x^2) dx = -\frac{1}{2\pi} \cos(\pi x^2) + C$

$$\int -\pi^2 x \cos(\pi x^2) dx = -\frac{\pi}{2} \sin(\pi x^2) + C$$

同步练习 P215 11

廖国勇 2016.4.18

11. 把第二类曲线积分 $\int_C Pdx + Qdy$ 化为第一类曲线积分, 其中

C 是抛物线 $y=x^2$ 上从点 $O(0,0)$ 到 $A(2,4)$ 的弧段.

解:
$$\begin{cases} x=x \\ y=x^2 \end{cases} \quad 0 \leq x \leq 2$$

$$T = (x', y') = (1, 2x)$$

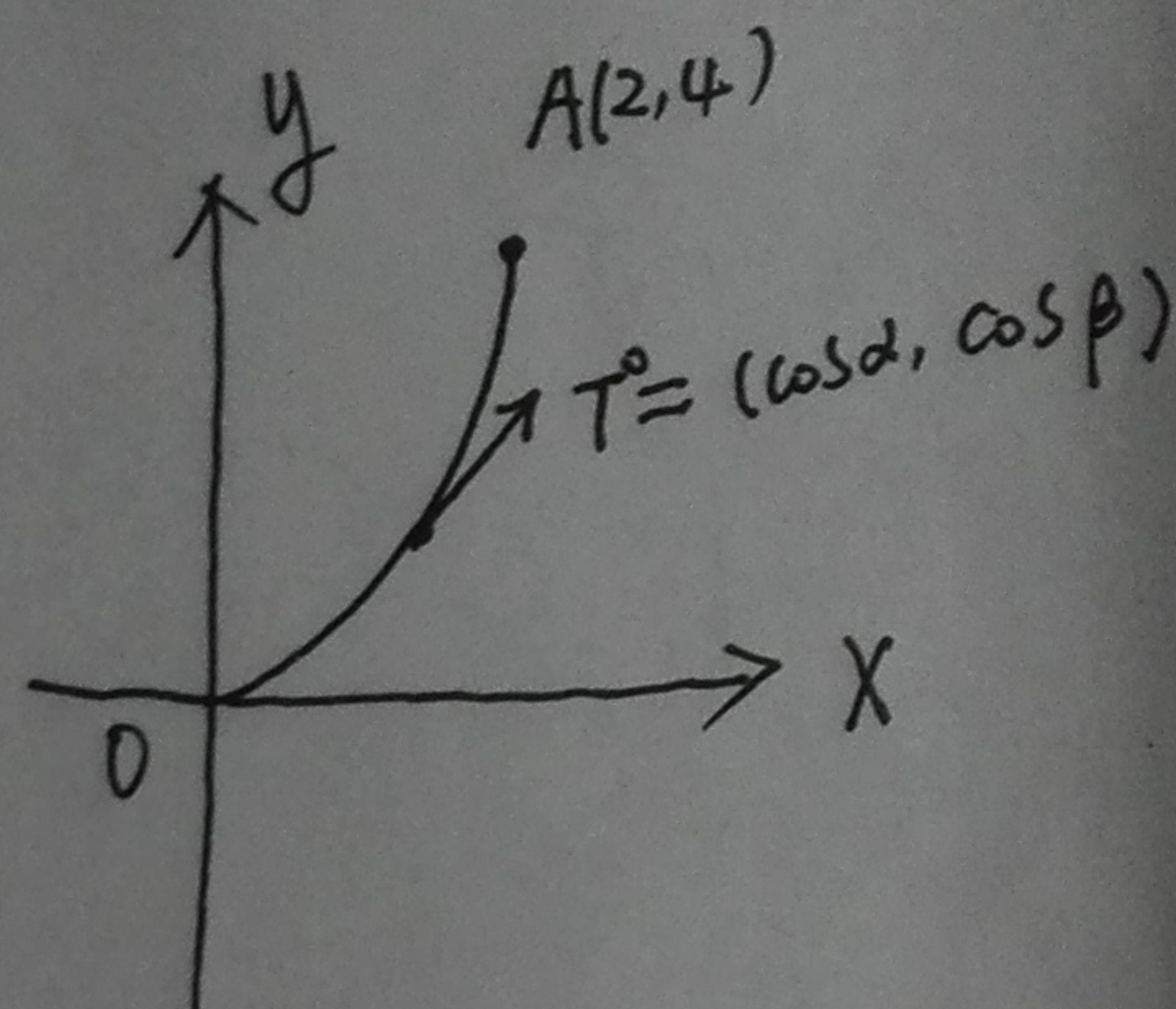
$$T^0 = \frac{T}{\|T\|} = \left(\frac{1}{\sqrt{1+4x^2}}, \frac{2x}{\sqrt{1+4x^2}} \right)$$

$$= (\cos \alpha, \cos \beta)$$

于是,
$$\int_C Pdx + Qdy = \int_C P \cos \alpha ds + Q \cos \beta ds$$

$$= \int_C \left(\frac{P}{\sqrt{1+4x^2}} + \frac{2x}{\sqrt{1+4x^2}} Q \right) ds$$

$$= \int_C \frac{P + 2xQ}{\sqrt{1+4x^2}} ds$$



同步练习 P246 12 廖国勇 2016.4.15

12. 设 L 是曲线 $x=t, y=t, z=t\sqrt{1-2t^2}$ 从对应于 $t=0$ 的点到 $t=\frac{\sqrt{2}}{2}$ 的点的弧段, 将第二类曲线积分 $\int_L p dx + Q dy + R dz$ 化为第一类曲线积分.

解: 曲线上任一点处切线的方向向量 $T = (x'_t, y'_t, z'_t)$

$$= (1, 1, \frac{2t}{\sqrt{1-2t^2}}) = (1, 1, \frac{2x}{\sqrt{1-2x^2}}) = (1, 1, \frac{2x}{1-z})$$

$$T^0 = \frac{T}{\|T\|} = \left(\frac{1}{\frac{\sqrt{2}}{1-z}}, \frac{1}{\frac{\sqrt{2}}{1-z}}, \frac{\frac{2x}{1-z}}{\frac{\sqrt{2}}{1-z}} \right)$$

$$= \left(\frac{1-z}{\sqrt{2}}, \frac{1-z}{\sqrt{2}}, \sqrt{2}x \right) = (\cos\alpha, \cos\beta, \cos\gamma)$$

于是, $\int_L p dx + Q dy + R dz = \int p \cos\alpha ds + Q \cos\beta ds + R \cos\gamma ds$

$$= \int_{\Gamma} \left(p \frac{1-z}{\sqrt{2}} + Q \frac{1-z}{\sqrt{2}} + R \sqrt{2}x \right) ds$$

$$= \int_{\Gamma} \frac{p(1-z) + Q(1-z) + R \cdot 2x}{\sqrt{2}} ds$$

同步练习 P246 13 廖国勇 2016.4.15

13. 求半径为2, 中心角为 $\frac{\pi}{3}$ 的匀圆弧($\rho=1$)的质心.

解: 以圆弧的圆心为原点, 以圆心与圆弧中点的连线为x轴由

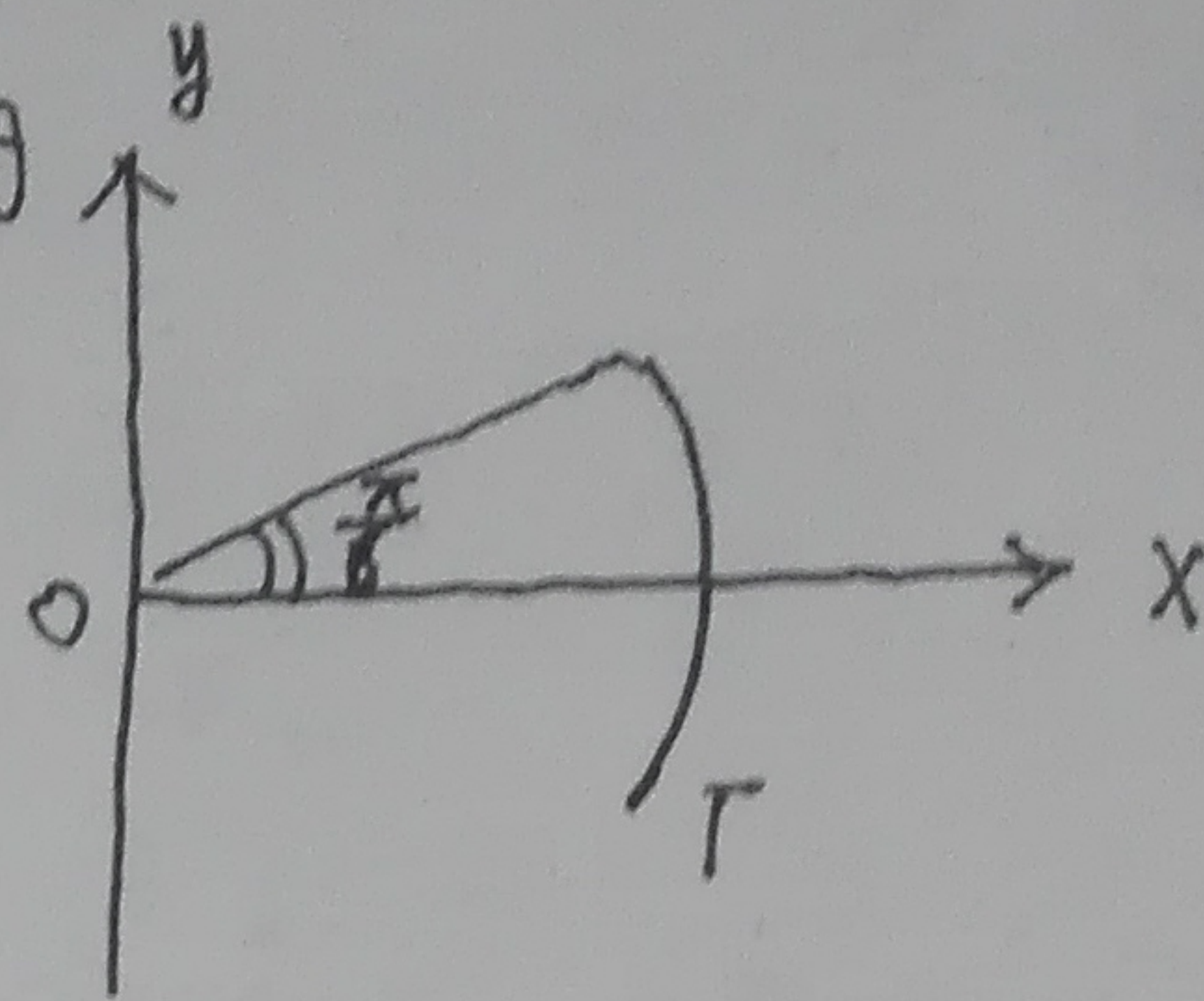
建立如右图所示的直角坐标系。 $x=2\cos\theta$, $y=2\sin\theta$

由对称性知, $\bar{y}=0$

$$\bar{x} = \frac{\int_{\Gamma} x ds}{\int_{\Gamma} ds} = \frac{\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} 2\cos\theta \cdot 2 d\theta}{\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} 2 d\theta}$$

$$= \frac{4\sin\theta \Big|_{-\frac{\pi}{6}}^{\frac{\pi}{6}}}{2\theta \Big|_{-\frac{\pi}{6}}^{\frac{\pi}{6}}} = \frac{4}{2 \cdot \frac{\pi}{3}} = \frac{6}{\pi}$$

即质心坐标为 $(\frac{6}{\pi}, 0)$



$$ds = \sqrt{(dx)^2 + (dy)^2} = \sqrt{(2\sin\theta)^2 + (2\cos\theta)^2} d\theta = 2 d\theta$$

$$-\frac{\pi}{6} \leq \theta \leq \frac{\pi}{6}$$

同步练习 P216 14 廖国勇 2016.4.15

14. 设空间曲线的方程 $x=2\cos t$, $y=2\sin t$, $z=3t$, 其中 $0 \leq t \leq 2\pi$, 且线密度 $\rho = x^2 + y^2 + z^2$, 求该曲线对于 z 轴的转动惯量.

解: $ds = \sqrt{(dx)^2 + (dy)^2 + (dz)^2} = \sqrt{(-2\sin t)^2 + (2\cos t)^2 + 3^2} dt = \sqrt{13} dt$

$$dI = d^2 \cdot dm = d^2 \cdot \rho ds$$

$$= (x^2 + y^2) \cdot (x^2 + y^2 + z^2) \cdot \sqrt{(dx)^2 + (dy)^2 + (dz)^2}$$

$$= 4 \cdot (4 + 9t^2) \cdot \sqrt{13} dt = (16 + 36t^2) \sqrt{13} dt$$

于是, $I = \int_0^{2\pi} (16 + 36t^2) \sqrt{13} dt$

$$= 32\sqrt{13}\pi + 96\sqrt{13}\pi^3 = (1 + 3\pi^2)32\sqrt{13}\pi$$